

On Homogeneous Integral Table Algebras

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1. INTRODUCTION

Integral table algebras abstract common features of character rings and centers of group algebras of finite groups, and of the Bose–Mesner algebras of commutative association schemes. In particular, every integral table algebra (ITA) has a distinguished basis, each element of which possesses a positive integer *degree*. We call such algebras *homogeneous* when the degrees of all the nonidentity basis elements are the same (Definition 1.3 below). We show in Theorem 1 that every ITA can be rescaled to one which is homogeneous. Thus, properties of homogeneous ITAs will apply to integral table algebras in general. We establish in this paper some results on homogeneous integral table algebras (Theorems 2, 3, 4, 5) which we hope will be generalized over time into a broad theory for these structures. The remainder of this Introduction is devoted to the statement of our main results, which are given as soon as we present the necessary definitions and notation. Throughout, \mathbb{C} denotes the complex numbers, \mathbb{R} the reals, \mathbb{R}^+ the positive reals, and \mathbb{Z}^+ the positive integers.

DEFINITION 1.1 [ABEF, B1, B2]. Let $\mathbf{B} = \{b_1, b_2, \dots, b_k\}$ be a basis of a finite dimensional, associative, and commutative algebra A over the complex field \mathbb{C} , with identity element $1_A = b_1$. Then (A, \mathbf{B}) is a *table algebra* (and \mathbf{B} is a *table basis*) if and only if the following hold:

(1.1)(I) For all i, j, m , $b_i b_j = \sum_{m=1}^k \beta_{ijm} b_m$, with β_{ijm} a nonnegative real number.

(1.1)(II) There is an algebra automorphism (denoted by $\bar{}$) of A whose order divides 2, such that $b_i \in \mathbf{B}$ implies that $\bar{b}_i \in \mathbf{B}$. (Then \bar{i} is defined by $b_{\bar{i}} = \bar{b}_i$, and $b_i \in \mathbf{B}$ is called *real* if $i = \bar{i}$.)

(1.1)(III) For all i, j , $\beta_{ij1} \neq 0$ if and only if $j = \bar{i}$.

This definition is equivalent to the one stated in [AB], because of [AB, Lemma 2.1]. Furthermore, [AB, Lemma 2.9] implies that there is an algebra homomorphism $f: A \rightarrow \mathbb{C}$ such that $f(b_i) = f(\bar{b}_i) \in \mathbb{R}^+$ for all i . Such a map f is uniquely determined by the orthogonality relations which hold for (A, \mathbf{B}) [BI, Theorem 5.5; B1, Proposition 2.11]. The positive real numbers $f(b_i)$, $1 \leq i \leq k$, are called the *degrees* of (A, \mathbf{B}) .

DEFINITION 1.2 [B1]. A table algebra (A, \mathbf{B}) is called *integral* iff all the structure constants β_{ijm} and all the degrees $f(b_i)$ are rational integers.

Any finite group G yields two examples of integral table algebras: $(Z(\mathbb{C}G), \text{Cla}(G))$, the center of the group algebra, with table basis the set of sums \hat{C} of G -conjugacy classes C , with automorphism $-$ extended linearly from inversion in G , and with degrees $f(\hat{C}) = |C|$ for all $\hat{C} \in \text{Cla}(G)$; and $(\text{Ch}(G), \text{Irr}(G))$, the ring of complex valued class functions on G , with table basis the set of irreducible characters of G , with automorphism $-$ extended linearly from complex conjugation of characters, and with degrees $f(\chi) = \chi(1)$ for all $\chi \in \text{Irr}(G)$. Another example is the Bose–Mesner algebra of a commutative association scheme, with table basis the set of adjacency matrices [BI, Sect. II.2].

DEFINITION 1.3. A table algebra (A, \mathbf{B}) is called *homogeneous* (of degree λ) iff $|\mathbf{B}| > 1$ and, for some fixed $\lambda \in \mathbb{R}^+$,

$$\text{degree } f(b) = \lambda \quad \text{for all } b \in \mathbf{B} \setminus \{1\}.$$

DEFINITION 1.4 [AB, B1]. Let (A, \mathbf{B}) be a table algebra, with $|\mathbf{B}| = k$. For $1 \leq i \leq k$, pick any $\gamma_i \in \mathbb{R}^+$, subject only to $\gamma_1 = 1$ and $\gamma_i = \gamma_i$ for all i . Let $b'_i = \gamma_i b_i$ and $\mathbf{B}' = \{b'_i | 1 \leq i \leq k\}$. Then it is easily seen that (A, \mathbf{B}') is also a table algebra, with respect to the same automorphism $-$. Any such \mathbf{B}' is called a *rescaling* of \mathbf{B} .

Remark 1.5. Let (A, \mathbf{B}) be any table algebra, and fix any $\lambda \in \mathbb{R}^+$. Let \mathbf{B}' be the rescaling of \mathbf{B} where $b'_i = (\lambda/f(b_i))b_i$ for all $i > 1$. Then $f(b'_i) = \lambda$ for all $i > 1$, so that (A, \mathbf{B}') is homogeneous of degree λ .

The remark shows that homogeneous table algebras are plentiful. The rescaling of any ITA to one which is homogeneous, while not quite so immediate as in the remark, is straightforward nonetheless, as our first theorem shows. This result also appears as a remark in [X, p. 764].

THEOREM 1. Any integral table algebra has a rescaling which is also integral, and which is homogeneous of some positive integer degree.

The degree of homogeneity which results when an ITA is rescaled to a homogeneous ITA is usually rather large, as may be seen from the proof of Theorem 1 in Section 2.

Let (A, \mathbf{B}) be a table algebra. A nonempty subset $\mathbf{C} \subseteq \mathbf{B}$ is called a *table subset* (or a *C-subset*) of \mathbf{B} iff $\text{Supp}_{\mathbf{B}}(b_i b_j) \subseteq \mathbf{C}$ for all $b_i, b_j \in \mathbf{C}$. Any table subset is stable under $-$ and contains 1_A [AB, Proposition 2.7; B1, Proposition 2.19]. For any $c \in \mathbf{B}$, the set \mathbf{B}_c defined by

$$\mathbf{B}_c := \bigcup_{n=1}^{\infty} \text{Supp}_{\mathbf{B}}(c^n)$$

is easily seen to be a table subset of \mathbf{B} , called the *table subset generated by c*. Another table subset given in terms of a fixed element $c \in \mathbf{B}$ is $\text{sta}_{\mathbf{B}}c$, called the *stabilizer of c in B* and defined as

$$\text{sta}_{\mathbf{B}}c := \{b \in \mathbf{B} | \text{Supp}_{\mathbf{B}}(bc) = \{c\}\}.$$

A table algebra (A, \mathbf{B}) is called *simple* iff \mathbf{B} has no table subset other than \mathbf{B} and $\{1\}$. Thus, (A, \mathbf{B}) is simple iff $\mathbf{B}_c = \mathbf{B}$ for all $c \in \mathbf{B} \setminus \{1\}$.

An element c of \mathbf{B} is called *faithful* iff $\mathbf{B}_c = \mathbf{B}$. (Note that for any finite group G , $\hat{C} \in \text{Cla}(G)$ is faithful iff $\langle C \rangle = G$, and $\chi \in \text{Irr}(G)$ is faithful iff χ is faithful in the usual character-theoretic sense.) Also, $c \in \mathbf{B}$ is called *linear* iff $\text{Supp}_{\mathbf{B}}(c^n) = \{1\}$ for some $n > 0$. This is equivalent to $\text{Supp}_{\mathbf{B}}(c\bar{c}) = \{1\}$ [AB, Proposition 3.2]. (Now $\hat{C} \in \text{Cla}(G)$ is linear iff $C \subseteq Z(G)$ iff $|C| = 1$, and $\chi \in \text{Irr}(G)$ is linear iff $\chi(1) = 1$.) A table subset of \mathbf{B} is called *abelian* iff each of its elements is linear. The set of all linear elements of \mathbf{B} , denoted $\mathbf{L}(\mathbf{B})$, is a table subset [AB, Proposition 3.2]. If H is any finite abelian group then $(\mathbb{C}H, H)$ is an abelian table algebra, and [AB, Theorem A] shows that any abelian table algebra is a rescaling of one of this form. Note that $(\mathbb{C}H, H)$ is a homogeneous ITA of degree 1.

Two table algebras (A, \mathbf{B}) and (A', \mathbf{B}') are called *isomorphic* (denoted $\mathbf{B} \cong \mathbf{B}'$) when there exists an algebra isomorphism $\psi: A \rightarrow A'$ such that $\psi(\mathbf{B})$ is a rescaling of \mathbf{B}' ; and the algebras are called *exactly isomorphic* (denoted $\mathbf{B} \cong_x \mathbf{B}'$) when $\psi(\mathbf{B}) = \mathbf{B}'$ [B1, Sect. 1]. So $\mathbf{B} \cong_x \mathbf{B}'$ means that \mathbf{B} and \mathbf{B}' yield the same structure constants.

Remark 1.6. (i) If (A, \mathbf{B}) is a homogeneous ITA of degree 1, then by [B1, Proposition 5.9], \mathbf{B} is abelian. Thus by [AB, Theorem A], $(A, \mathbf{B}) \cong_x (\mathbb{C}H, H)$ for some finite abelian group H .

(ii) The homogeneous ITAs of degree 2 which contain a faithful element are classified in [B3]. The proof is based on [B2, Theorem 2], which finds all ITAs with a faithful, real element of degree 2. The table bases $\mathbf{O}(2, \mathbb{Z}_{2n+1})$ and \mathbf{D}_{2n} of Examples 1.16 and 1.17 below are instances of homogeneous ITAs of degree 2. Work in progress of the authors [BX], and

of Arad, Fisman, Miloslavsky, and Muzychuk [AFMM], goes quite a way toward classifying the homogeneous ITAs of degree 3 with a faithful element. Theorems 2, 3, and 5 and Example 3.3 of the present paper are applied in the study of homogeneous ITAs of degree 3.

For any table algebra (A, \mathbf{B}) , there is a positive definite Hermitian form $(\ , \)$ on A with the following properties [B1, Propositions 2.4, 2.5]: \mathbf{B} is an orthogonal set with respect to $(\ , \)$; for all $b \in \mathbb{R}\mathbf{B}$ (the real span of \mathbf{B}) and all $a, c \in A$,

$$(ab, c) = (a, \bar{b}c); \quad (1.7)$$

and for all $b_i \in \mathbf{B}$, and β_{iii} the structure constant as in Definition 1.1,

$$(b_i, b_i) = \beta_{iii}. \quad (1.8)$$

It follows that, for all $a, b, c \in \mathbf{B}$,

$$c \in \text{Supp}_{\mathbf{B}}(ab) \Leftrightarrow a \in \text{Supp}_{\mathbf{B}}(\bar{b}c). \quad (1.9)$$

DEFINITION 1.10. Let (A, \mathbf{B}) be a table algebra and $b \in \mathbf{B}$. The *length* of b is defined as the positive real number (b, b) . Furthermore, b is called *standard* iff $(b, b) = f(b)$.

Each $\hat{C} \in \text{Cla}(G)$, where G is any finite group, is standard. Any table basis of a table algebra has a rescaling in which every element is standard [B1, Proposition 2.1].

PROPOSITION 1.11. Let (A, \mathbf{B}) be an ITA which is homogeneous of degree λ , for some $\lambda \in \mathbb{Z}^+$. Then for all $b_i \in \mathbf{B} \setminus \{1\}$, $(b_i, b_i) = \mu_i \lambda$ for some integer μ_i with $1 \leq \mu_i \leq \lambda$.

The easy proof of this proposition is deferred until Section 2.

Remark 1.12. Note that in the situation of Proposition 1.11, $\mu_i = 1$ iff b_i is standard, and $\mu_i = \lambda$ iff b_i is linear.

THEOREM 2. Let (A, \mathbf{B}) be an integral table algebra which is homogeneous of degree λ . Assume that $\mathbf{B} \setminus \{1\}$ contains a standard element. If $b_i \in \mathbf{B}$ is nonlinear, and of length $(b_i, b_i) = \mu_i \lambda$, then $\mu_i < (3 - \sqrt{6})\lambda$.

Remark 1.13. (i) Examples 3.1 and 3.2 below show that many values of $\mu_i \leq \lambda/2$ occur, including $\mu_i = \lambda/2$ and $\mu_i \nmid \lambda$.

(ii) If (A, \mathbf{B}) is as in Theorem 2, $b_i \in \mathbf{B} \setminus \{1\}$ is standard, $b_i \in \mathbf{B}$ is nonlinear, and $|\text{Supp}_{\mathbf{B}}(b_i b_i)| = N$, then the proof of Theorem 2 shows (as noted in Section 2) that $\mu_i \leq (N/(2N - 1))\lambda$. The inequality (2.13) below displays another way in which the best possible upper bound on μ_i tends

asymptotically to $\lambda/2$. On the other hand, the authors know of no examples where $\lambda/2 < \mu_i < \lambda$. This raises the following open question: Can the bound in Theorem 2 be improved to $\mu_i \leq \lambda/2$?

(iii) The authors are grateful to the referee, who pointed out that our original estimate ($\mu_i < (2 - \sqrt{2})\lambda$) could be improved to the bound as stated in Theorem 2, and who indicated how our original proof could be expanded, after (2.12), to achieve this improvement. The current ending of our proof, including (2.13), is a slight generalization of the referee's suggestions.

Next, we give further examples of homogeneous table algebras which do not arise from rescaling as in Remark 1.5 or Theorem 1.

EXAMPLE 1.14. Let H be a finite abelian group which admits a fixed-point-free action by the cyclic group \mathbb{Z}_n , for some $n > 0$. Let $G = \mathbb{Z}_n \ltimes H$, and

$$\mathbf{O} = \mathbf{O}(n, H) := \{\hat{C} \in \text{Cla}(G) \mid C \subseteq H\},$$

the set of sums over the orbits of \mathbb{Z}_n on H . Then \mathbf{O} is a table subset of $\text{Cla}(G)$. For $A = \langle \mathbf{O} \rangle$, (A, \mathbf{O}) is an ITA which is homogeneous of degree n . In the special case $\mathbf{O}(2, \mathbb{Z}_{2n+1})$, G is the dihedral group of order $2(2n+1)$.

EXAMPLE 1.15. Let G be the dihedral group of order $4n$ for some $n \in \mathbb{Z}^+$, and let $\mathbb{Z}_{2n} \cong H \triangleleft G$. Let $\mathbf{B} = \{\hat{C} \in \text{Cla}(G) \mid C \subseteq H\}$. Thus for $H = \langle x \rangle$, $\mathbf{B} = \{1, c_1, c_2, \dots, c_n\}$, where $c_i = x^i + x^{-i}$ for $1 \leq i < n$ and $c_n = x^n$. Define $\mathbf{D}_{2n} := \{1, c_1, c_2, \dots, c_{n-1}, 2c_n\}$, a rescaling of \mathbf{B} , and let $A = \langle \mathbf{D}_{2n} \rangle$. Then (A, \mathbf{D}_{2n}) is a homogeneous ITA of degree 2. See [B2, p. 910], where this example realizes the affine diagram \widehat{BC}_n as an ITA.

EXAMPLE 1.16. Let (A, \mathbf{B}) be a table algebra which is homogeneous of degree $\lambda \in \mathbb{R}^+$. Let H be an abelian group and $C := \mathbb{C}H$. Then $A \otimes_{\mathbb{C}} C$, with basis $\mathbf{B} \otimes H := \{b \otimes h \mid b \in \mathbf{B}, h \in H\}$ is a table algebra, where $\overline{b \otimes h} = \overline{b} \otimes h^{-1}$ and $f(b \otimes h) = f(b)$ for all $b \in \mathbf{B}, h \in H$ [B1, Example 1.5]. Define

$$\begin{aligned} (\mathbf{B} \otimes H)' &:= \{b \otimes h \mid b \in \mathbf{B} \setminus \{1\}, h \in H\} \\ &\cup \{1 \otimes \lambda h \mid h \in H \setminus \{1\}\} \cup \{1 \otimes 1\}, \end{aligned}$$

a rescaling of $\mathbf{B} \otimes H$. Then it is easily seen that $(A \otimes C, (\mathbf{B} \otimes H)')$ is homogeneous of degree λ ; and if (A, \mathbf{B}) is an ITA then so is $(A \otimes C, (\mathbf{B} \otimes H)')$. We may regard $(\{1\} \otimes H)'$ as simply $H' := \{1\} \cup \{\lambda h \mid h \in H \setminus \{1\}\}$.

EXAMPLE 1.17. We present two families of homogeneous table algebras of degree λ , with bases denoted $\mathbf{T}_0(\lambda)$ for $\lambda > 1$, and $\mathbf{V}(3, \lambda)$ for $\lambda \geq 2$. It is easy to check that the given products, extended bilinearly, yield table algebras which are homogeneous of degree λ , and are integral when $\lambda \in \mathbb{Z}^+$. Furthermore, $\mathbf{T}_0(\lambda)$ is the smallest member of the family $\mathbf{T}_n(\lambda)$ defined below in Example 3.3 and Definition 3.4, and $\mathbf{V}(3, \lambda)$ is the special case $\mathbf{V}(3, \lambda, 1, \lambda - 1)$ of the family $\mathbf{V}(3, \lambda, \mu, \beta)$ given below in Example 3.2.

$$\mathbf{T}_0(\lambda) := \{1, b\}, \quad \text{with } b^2 = \lambda \cdot 1 + (\lambda - 1)b;$$

$$\mathbf{V}(3, \lambda) := \{1, b, c\}, \quad \text{with } b^2 = \lambda \cdot 1 + (\lambda - 1)c, \quad bc = (\lambda - 1)b + c, \\ c^2 = \lambda \cdot 1 + b + (\lambda - 2)c.$$

Note that $\mathbf{T}_0(2) \cong_x \mathbf{O}(2, \mathbb{Z}_3)$ and $\mathbf{V}(3, 2) \cong_x \mathbf{O}(2, \mathbb{Z}_5)$.

DEFINITION 1.18. Let (A, \mathbf{B}) be a table algebra and $b \in \mathbf{B}$. The *width* of b is defined as $|\text{Supp}_{\mathbf{B}}(b\bar{b})|$.

Thus, “width” is always a positive integer. Note that our terminology differs from that in [X], where our notion of “width” is there called “length.” Note also that b has width 1 iff b is linear.

The next theorem extends a result of one of the authors [X, Theorem 1] which contains the additional hypothesis that the given faithful element is real.

THEOREM 3. Let (A, \mathbf{B}) be an integral table algebra which is homogeneous of degree λ for some $\lambda > 2$, and is such that \mathbf{B} contains a standard, faithful element of width 2. Then \mathbf{B} is exactly isomorphic to $(\mathbf{T}_0(\lambda) \otimes \mathbb{Z}_m)'$ or $(\mathbf{V}(3, \lambda) \otimes \mathbb{Z}_m)'$ for some $m \in \mathbb{Z}^+$.

Let (A, \mathbf{B}) be a table algebra and \mathbf{C} a table subset of \mathbf{B} . There is an idempotent, denoted $e_{\mathbf{C}}$, which equals a positive real scalar times $\sum_{b_i \in \mathbf{C}} (f(b_i)/\beta_{i1})b_i$ [B1, Corollary 3.13]. Also, $\{\text{Supp}_{\mathbf{B}}(e_{\mathbf{C}}b_i) | b_i \in \mathbf{B}\}$ partitions \mathbf{B} into disjoint classes [B1, Theorem 1, (1*)].

DEFINITION 1.19. Let $A, \mathbf{B}, \mathbf{C}$, and $e := e_{\mathbf{C}}$ be as above. Define

$$\mathbf{B}/\mathbf{C} := \{e\} \cup \{eb_i | b_i \in \mathbf{B} \setminus \mathbf{C} \text{ and } f(b_i) \leq f(b_j) \text{ for all } b_j \in \text{Supp}_{\mathbf{B}}(eb_i)\}.$$

It follows from [B1, Theorems 1, 2] that $(Ae, \mathbf{B}/\mathbf{C})$ is a table algebra, is an epimorphic image of (A, \mathbf{B}) (as defined in [B1, Definition 1.8]), and up to isomorphism is the unique such image where \mathbf{C} comprises exactly those elements of \mathbf{B} which map to positive scalar multiples of the identity. Then $(Ae, \mathbf{B}/\mathbf{C})$ is called the *quotient table algebra determined by \mathbf{C}* . The positive-valued homomorphism on \mathbf{B}/\mathbf{C} is just the restriction of f .

Remark 1.20. Let (A, \mathbf{B}) be an ITA which is homogeneous of degree λ . Let \mathbf{C} be a table subset of \mathbf{B} . It follows from [B1, Lemma 5.4] that $(Ae_{\mathbf{C}}, \mathbf{B}/\mathbf{C})$ is again an ITA which is homogeneous of degree λ .

Composition series are defined in the obvious way for any table basis, and a Jordan–Holder theorem holds for such chains of table subsets [B1, Theorem 5]. A table algebra (A, \mathbf{B}) is called *nilpotent* iff every composition factor of \mathbf{B} is abelian [B1, Definition 1.16].

DEFINITION 1.21. The *upper central series* of a table algebra (A, \mathbf{B}) is the chain of table subsets $\mathbf{L}^{(i)}(\mathbf{B})$ for all $i \geq 0$, defined as follows: $\mathbf{L}^{(0)}(\mathbf{B}) := \{1\}$, $\mathbf{L}^{(1)}(\mathbf{B}) := \mathbf{L}(\mathbf{B})$, and recursively, $\mathbf{L}^{(i+1)}(\mathbf{B})$ is the table subset of \mathbf{B} such that $\mathbf{L}^{(i+1)}(\mathbf{B})/\mathbf{L}^{(i)}(\mathbf{B}) = \mathbf{L}(\mathbf{B}/\mathbf{L}^{(i)}(\mathbf{B}))$ (see [B1, Theorem 3]).

Thus, each $\mathbf{L}^{(i)}(\mathbf{B})$ is nilpotent, and for some n , and all $j \geq 0$,

$$\mathbf{L}^{(0)}(\mathbf{B}) \subseteq \mathbf{L}^{(1)}(\mathbf{B}) \subseteq \mathbf{L}^{(2)}(\mathbf{B}) \subseteq \cdots \subseteq \mathbf{L}^{(n)}(\mathbf{B}) = \mathbf{L}^{(n+j)}(\mathbf{B}).$$

If there is no ambiguity, we abbreviate $\mathbf{L}^{(i)}(\mathbf{B})$ as $\mathbf{L}^{(i)}$. (Note the analogous definition of $\mathbf{L}_p^{(i)}(\mathbf{B})$, p a prime, in [B1, Sect. 5]). When (A, \mathbf{B}) is $(Z(\mathbb{C}G), \text{Cla}(G))$ for a finite group G , then the $\mathbf{L}^{(i)}$ correspond to the usual upper central series of G . That is,

$$\mathbf{L} = \{\hat{C} \in \text{Cla}(G) \mid C \subseteq Z(G)\},$$

$$\mathbf{L}^{(2)} = \{\hat{C} \in \text{Cla}(G) \mid C \subseteq N, \text{ where } N/Z(G) = Z(G/Z(G))\},$$

and so forth.

For any table algebra (A, \mathbf{B}) , it is easy to verify that the central series term $\mathbf{L}^{(n)}$ with $\mathbf{L}^{(n)} = \mathbf{L}^{(n+1)}$ coincides with the *nilpotent radical* \mathbf{B}^{nil} , the unique maximal nilpotent table subset of \mathbf{B} (as in [B1, Theorem 7]). In particular, the following result holds.

PROPOSITION 1.22. *A table algebra (A, \mathbf{B}) is nilpotent iff $\mathbf{L}^{(n)}(\mathbf{B}) = \mathbf{B}$ for some $n > 0$.*

DEFINITION 1.23. If (A, \mathbf{B}) is a nilpotent table algebra, then the minimal $n > 0$ such that $\mathbf{L}^{(n)}(\mathbf{B}) = \mathbf{B}$ is called the *nilpotence class* of \mathbf{B} .

Remark 1.24. One may define a *lower central series* for any table algebra (A, \mathbf{B}) as follows: $\mathbf{C}^{(0)}(\mathbf{B}) = \mathbf{B}$, and given $\mathbf{C}^{(i)}(\mathbf{B})$, set $\mathbf{C}^{(i+1)}(\mathbf{B})$ equal to the intersection of all table subsets of \mathbf{B} which contain $\text{Supp}_{\mathbf{B}}(bb)$, for all $b \in \mathbf{C}^{(i)}(\mathbf{B})$. Then $\mathbf{C}^{(i+1)}(\mathbf{B}) \subseteq \mathbf{C}^{(i)}(\mathbf{B})$, and it is easy to see that the series terminates in $\{1\}$ iff \mathbf{B} is nilpotent. Furthermore, the minimal n such that $\mathbf{C}^{(n)}(\mathbf{B}) = \{1\}$ equals the nilpotence class of \mathbf{B} .

The next theorem, for nilpotent homogeneous integral table algebras, stands in contrast to Example 3.2 below.

THEOREM 4. *Suppose that (A, \mathbf{B}) is a nilpotent, homogeneous ITA of degree λ , with nilpotence class n . Let $b \in \mathbf{B}$ with length $(b, b) = \mu\lambda$. Then*

$$\lambda = \mu \cdot |\text{sta}_{\mathbf{L}}(b)| \cdot |\text{sta}_{\mathbf{L}^{(2)}/\mathbf{L}}(be_{\mathbf{L}})| \cdot |\text{sta}_{\mathbf{L}^{(3)}/\mathbf{L}^{(2)}}(be_{\mathbf{L}^{(2)}})| \cdots |\text{sta}_{\mathbf{L}^{(n)}/\mathbf{L}^{(n-1)}}(be_{\mathbf{L}^{(n-1)}})|.$$

In particular, $\mu|\lambda$.

The final main result of this paper, Theorem 5 below, concerns homogeneous ITAs of prime degree. It is preceded by two classes of examples. Example 1.25 establishes some context, and Example 1.26 describes algebras which occur in the conclusion of the theorem.

EXAMPLE 1.25. Let p be a prime, H a finite abelian group acted upon by $\mathbb{Z}_p = \langle \phi \rangle$, but not necessarily fixed-point freely. Each \mathbb{Z}_p -orbit in H has size 1 or p . Let $G = \mathbb{Z}_p \rtimes H$ and $\mathbf{S} = \{\hat{C} \in \text{Cla}(G) | C \subseteq H\}$. Then \mathbf{S} is a table subset of $\text{Cla}(G)$ such that $|C| = p$ or 1 for all $\hat{C} \in \mathbf{S}$. Let \mathbf{B} be the rescaling of \mathbf{S} which consists of 1, each $\hat{C} \in \mathbf{S}$ with $|C| = p$, and $p\hat{C}$ for each $\hat{C} \neq 1$ in \mathbf{S} with $|C| = 1$. Define $\mathbf{O}(p, H)' := \mathbf{B}$. If $\hat{C}, \hat{D}, \hat{U}$ are in \mathbf{S} with $|C| = |D| = p$, $U = \{u\}$, and $\hat{U} \in \text{Supp}(\hat{C}\hat{D})$, then there exist $c \in C$, $d \in D$ with $cd = u$. Hence, $u = u^{\phi^i} = c^{\phi^i}d^{\phi^i}$ for all i , and the coefficient with which \hat{U} appears in $\hat{C}\hat{D}$ is p . It follows that $\mathbf{O}(p, H)'$ is an ITA which is homogeneous of degree p . If H is a p -group, then G is nilpotent and thus $\mathbf{O}(p, H)'$ is a nilpotent table algebra.

Two special cases of $\mathbf{O}(p, H)'$ deserve mention. First, let $p = 2$, $H = \mathbb{Z}_{2^n}$ for some $n > 0$, where \mathbb{Z}_2 acts on H by inversion. Then G is the dihedral group of order 2^{n+1} , and $\mathbf{O}(2, \mathbb{Z}_{2^n})' = \mathbf{D}_{2^n}$ as in Example 1.15. Note that \mathbf{D}_{2^n} has nilpotence class n , and that every element is real.

For the second special case, let $H = \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ (p times), and let $\mathbb{Z}_p = \langle \phi \rangle$ act on H by cyclic permutation of coordinates. In other words, G is the wreath product $\mathbb{Z}_p \wr \mathbb{Z}_p$. Let $\mathbf{B} = \mathbf{O}(p, H)'$. It is not hard to check that

$$\mathbf{L}(\mathbf{B}) = \{(1, 1, \dots, 1)\} \cup \{p(x, x, \dots, x) | x \in \mathbb{Z}_p \setminus \{1\}\},$$

\mathbf{B} has nilpotence class p , and any element in $\mathbf{B} \setminus \mathbf{L}^{(p-1)}(\mathbf{B})$ is faithful.

EXAMPLE 1.26. We present two families of nilpotent, homogeneous ITAs of degree λ (where λ is a fixed positive integer), $\mathbf{N}((2), \lambda)$ and $\mathbf{N}((3), \lambda)$, with nilpotence classes 2 and 3, resp. First, define, as the basis

for a complex vector space A ,

$$\mathbf{N}((2), \lambda) := \{1, \lambda u, \lambda u^2, \dots, \lambda u^{\lambda-1}, b\},$$

where $\langle u \rangle = \mathbb{Z}_\lambda$, a cyclic group of order λ , $\lambda u^i \cdot b = \lambda b = b \cdot \lambda u^i$ for all i , and $b^2 = \lambda(1 + u + u^2 + \dots + u^{\lambda-1})$. It is easy to check that these products extend to a bilinear multiplication on A which is associative and commutative. Then $(A, \mathbf{N}((2), \lambda))$ is an ITA, with $\overline{u^i} = u^{-i}$ and $\overline{b} = b$; and $\mathbf{N}((2), \lambda)$ is homogeneous of degree λ . Note that $\mathbf{L}(\mathbf{N}((2), \lambda)) = \{1, \lambda u, \dots, \lambda u^{\lambda-1}\}$, and $\mathbf{L}^{(2)}(\mathbf{N}((2), \lambda)) = \mathbf{N}((2), \lambda)$. Next, we define

$$\mathbf{N}((3), \lambda)$$

$$:= \{1, \lambda u, \lambda u^2, \dots, \lambda u^{\lambda-1}, v_1, v_2, \dots, v_{\lambda-1}, b, ub, u^2b, \dots, u^{\lambda-1}b\},$$

as a basis for a complex vector space A of dimension $3\lambda - 1$. We define multiplication (with $u^i b$ already given) so that $\langle u \rangle = \mathbb{Z}_\lambda$, and

$$v_i v_j = \begin{cases} \lambda v_{i+j} \text{ (read } i+j \text{ mod } \lambda) & \text{if } i+j \neq \lambda; \\ \lambda \cdot \sum_{i=0}^{\lambda-1} u^i & \text{if } i+j = \lambda; \end{cases}$$

$$u^i v_j = v_j u^i = v_j \quad \text{for all } i, j;$$

$$b^2 = \lambda 1 + \sum_{j=1}^{\lambda-1} v_j;$$

$$v_j b = b v_j = b \cdot \sum_{i=0}^{\lambda-1} u^i, \quad \text{for all } j.$$

This extends to a multiplication in A which is associative and commutative. Direct verification of associativity is tedious, but an alternative construction of these algebras is given in Proposition 3.7 below. Then $(A, \mathbf{N}((3), \lambda))$ is an ITA which is homogeneous of degree λ , where $\overline{u^i} = u^{-i}$, $\overline{v_i} = v_{\lambda-i}$, $\overline{b} = b$. Observe that $\mathbf{L}(\mathbf{N}((3), \lambda)) = \{1, \lambda u, \dots, \lambda u^{\lambda-1}\}$, $\mathbf{L}^{(2)}(\mathbf{N}((3), \lambda)) = \{1, \lambda u, \dots, \lambda u^{\lambda-1}, v_1, v_2, \dots, v_{\lambda-1}\}$, and $\mathbf{L}^{(3)}(\mathbf{N}((3), \lambda)) = \mathbf{N}((3), \lambda)$.

THEOREM 5. *Suppose that (A, \mathbf{B}) is a nilpotent, homogeneous integral table algebra of degree λ , where λ is an odd prime. Assume also that \mathbf{B} has a real, faithful element. Then \mathbf{B} is exactly isomorphic to one of $\mathbb{Z}'_2, \mathbf{N}((2), \lambda)$, or $\mathbf{N}((3), \lambda)$. In particular, the nilpotence class of \mathbf{B} is at most three.*

Remark 1.27. (i) If (A, \mathbf{B}) satisfies the hypotheses of Theorem 5, but with $\lambda = 2$, then there is no bound on the nilpotence class of \mathbf{B} . This is shown by the algebras \mathbf{D}_{2^n} from Example 1.25.

(ii) Suppose that (A, \mathbf{B}) is a nilpotent, homogeneous ITA of odd prime degree λ , and that \mathbf{B} has a (nonreal) faithful element. Then \mathbf{B} can have nilpotence class λ , as in the algebra $\mathbf{O}(\lambda, H)'$ of Example 1.25, where $G = \mathbb{Z}_\lambda \setminus \mathbb{Z}_\lambda$. It seems an open question as to whether the nilpotence class of such (A, \mathbf{B}) is always bounded above by some function of λ .

The rest of the paper is organized as follows. Section 2 contains the proofs of Theorems 1 and 2, and of Proposition 1.11. Section 3 presents a detailed description of Example 1.26 and of several other examples, including the family $\mathbf{T}(n, \lambda, m)$ of homogeneous ITAs of degree λ . This family will play an important role in subsequent work. Theorem 3 is proved in Section 4 and Theorems 4 and 5 are established in Section 5. Section 6 displays representation graphs, as defined in [B2, Section 5], for $\mathbf{T}_n(\lambda) = \mathbf{T}(n, \lambda, 1)$ when n is even, and for $\mathbf{N}((3), 5)$.

2. PROOFS OF THEOREMS 1 AND 2

Proof of Theorem 1. Let (A, \mathbf{B}) be an ITA, with structure constants β_{ijm} as in Definition 1.1. If $\mathbf{B}' = \{\gamma_i b_i | b_i \in \mathbf{B}\}$ is any rescaling of \mathbf{B} , with structure constants β'_{ijm} , then it is easy to see that

$$\beta'_{ijm} = (\gamma_i \gamma_j / \gamma_m) \beta_{ijm}, \quad \text{for all } i, j, m. \quad (2.1)$$

Now let $\gamma = \text{lcm}_i f(b_i)$, $\gamma_i = \gamma^2 / f(b_i)$ for each $i > 1$, $\gamma_1 = 1$, and $\mathbf{B}' = \{b'_i\} = \{\gamma_i b_i | b_i \in \mathbf{B}\}$. Then \mathbf{B}' is a rescaling of \mathbf{B} , since $f(b'_i) = f(b_i)$ for all i . Each $b'_i \in \mathbf{B}' \setminus \{1\}$ has degree $f(b'_i) = \gamma_i f(b_i) = (\gamma^2 / f(b_i)) \cdot f(b_i) = \gamma^2$. Thus, (A, \mathbf{B}') is homogeneous of degree $\gamma^2 \in \mathbb{Z}^+$. Furthermore, for all $i, j, m > 1$, (2.1) and the definition of γ yield

$$\begin{aligned} \beta'_{ijm} &= (\gamma^2 / f(b_i)) (\gamma^2 / f(b_j)) (f(b_m) / \gamma^2) \beta_{ijm} \\ &= (\gamma / f(b_i)) (\gamma / f(b_j)) f(b_m) \beta_{ijm} \in \mathbb{Z}^+ \cup \{0\}, \end{aligned}$$

and $\beta'_{ij1} = \delta_{ij} (\gamma^2 / f(b_i))^2 \beta_{i\bar{i}1} \in \mathbb{Z}^+ \cup \{0\}$. So (A, \mathbf{B}') is an ITA, and the theorem is proved.

Proof of Proposition 1.11. Let (A, \mathbf{B}) be a homogeneous ITA of degree $\lambda \in \mathbb{Z}^+$. For each $b_i \in \mathbf{B} \setminus \{1\}$, we apply the algebra homomorphism f to the equation

$$b_i \bar{b}_i = \sum_{m=1}^k \beta_{i\bar{i}m} b_m = (b_i, b_i) 1 + \sum_{m>1} \beta_{i\bar{i}m} b_m$$

to obtain

$$\lambda^2 = f(b_i) f(\bar{b}_i) = f(b_i \bar{b}_i) = (b_i, b_i) + \sum_{m>1} \beta_{i\bar{i}m} \lambda.$$

Each $\beta_{i\bar{i}m}$ a nonnegative integer thus implies that $\lambda(b_i, b_i)$ and that $\lambda^2 \geq (b_i, b_i) > 0$. The result follows.

LEMMA 2.2. *Let (A, \mathbf{B}) be a homogeneous ITA of degree λ . Then for all indices i, j, m with $m > 1$, $\beta_{ijm} \leq \lambda$.*

Proof. Apply the homomorphism f to the equation

$$b_i b_j = \beta_{ijm} b_m + \sum_{t \neq m} \beta_{ijt} b_t, \quad \beta_{ijt} \in \mathbb{Z}^+ \cup \{0\}.$$

We obtain $\lambda^2 = \beta_{ijm} \lambda + \sum_{t \neq m} \beta_{ijt} f(b_t) \geq \beta_{ijm} \lambda$. So $\lambda \geq \beta_{ijm}$.

Proof of Theorem 2. Let (A, \mathbf{B}) be a homogeneous ITA of degree λ . Let $b_i \in \mathbf{B}$ be nonlinear, and with μ_i maximal among nonlinear basis elements. Then $\mu := \mu_i < \lambda$ (Remark 1.12). We may assume that $\mu > \frac{1}{2}\lambda$. Then $\lambda \geq 3$ (Remark 1.6).

By hypothesis, there is a standard element $b_t \in \mathbf{B} \setminus \{1\}$, so that $\mu_t = 1$. Then for S the set of indices for $\text{Supp}_{\mathbf{B}}(b_i b_t)$,

$$b_i b_t = \sum_{j \in S} \beta_{itj} b_j, \quad \beta_{itj} \in \mathbb{Z}^+. \quad (2.3)$$

Now $\mu_t < \mu$ implies that $b_t \neq \bar{b}_i$, and thus $1 \notin S$ by (1.1)(III).

Since \mathbf{B} is orthogonal with respect to the Hermitian form $(\ , \)$, we have $\beta_{itj} \mu_j \lambda = (b_i b_t, b_j)$, $\beta_{tji} \mu \lambda = (b_i b_j, b_t)$, and $\beta_{j\bar{i}t} \mu_t \lambda = (b_j b_i, b_t)$. Then by (1.7),

$$0 < \beta_{itj} \mu_j = \beta_{tji} \mu = \beta_{j\bar{i}t}, \quad \text{for all } j \in S. \quad (2.4)$$

Now by (2.4) and Lemma 2.2, $\lambda \geq \beta_{tji} \mu > \beta_{j\bar{i}t} \lambda / 2$, which forces

$$\beta_{tji} = 1, \quad \text{for all } j \in S. \quad (2.5)$$

Hence, again by (2.4),

$$\beta_{j\bar{i}t} = \mu = \beta_{itj} \mu_j, \quad \text{for all } j \in S. \quad (2.6)$$

Therefore, $\mu_j | \mu$ for all $j \in S$, and (2.3) may be rewritten as

$$b_i b_t = \sum_{j \in S} \frac{\mu}{\mu_j} b_j. \quad (2.7)$$

The application of f to each side of (2.7) implies that $\lambda^2 = \sum_{j \in S} (\mu/\mu_j)\lambda$, and thus

$$\lambda = \sum_{j \in S} \mu/\mu_j. \quad (2.8)$$

Now for all $j \in S$, (2.6) yields

$$b_{\bar{j}}b_i = \mu b_{\bar{i}} + \sum_{s \neq \bar{i}} \beta_{jis} b_s. \quad (2.9)$$

If $i \in S$, (2.9) with $j = i$ implies that

$$b_{\bar{i}}b_i = \mu b_{\bar{i}} + \mu\lambda 1 + \sum_{s \neq \bar{i}, 1} \beta_{iis} b_s,$$

and thus

$$\lambda^2 = f(b_{\bar{i}}b_i) = \mu\lambda + \mu\lambda + \lambda \sum_{s \neq \bar{i}, 1} \beta_{iis} \geq 2\mu\lambda.$$

This forces $\mu \leq \lambda/2$, a contradiction. So $i \notin S$. Then for all $j \in S$, $j \neq i$ and (2.9) imply

$$\lambda^2 = f(b_{\bar{j}}b_i) = \mu\lambda + \lambda \sum_{s \neq \bar{i}, 1} \beta_{jis},$$

and so

$$\sum_{s \neq \bar{i}, 1} \beta_{jis} = \lambda - \mu, \quad \text{for all } j \in S. \quad (2.10)$$

Now $b_i\bar{b}_{\bar{i}} = \mu\lambda 1 + x$ and $b_j\bar{b}_{\bar{j}} = \mu_j\lambda 1 + y_j$, where x and y_j are nonnegative combinations of $\mathbf{B} \setminus \{1\}$, for all $j \in S$. Therefore, by (1.7) and (2.9),

$$\begin{aligned} \mu\mu_j\lambda^2 &\leq (\mu\lambda 1 + x, \mu_j\lambda 1 + y_j) = (b_i\bar{b}_{\bar{i}}, b_j\bar{b}_{\bar{j}}) = (b_i\bar{b}_{\bar{j}}, b_i\bar{b}_{\bar{i}}) \\ &= \mu^2(b_{\bar{i}}, b_{\bar{i}}) + \sum_{s \neq \bar{i}, 1} \beta_{jis}^2(b_s, b_s) = \mu^2\lambda + \sum_{s \neq \bar{i}, 1} \beta_{jis}^2 \mu_s \lambda. \end{aligned}$$

Hence,

$$\mu\mu_j\lambda \leq \mu^2 + \sum_{s \neq \bar{i}, 1} \beta_{jis}^2 \mu_s, \quad \text{for all } j \in S. \quad (2.11)$$

Now by (1.7),

$$\beta_{jis} \mu_s \lambda = (b_{\bar{j}}b_i, b_s) = (b_{\bar{j}}, b_{\bar{i}}b_s) = \beta_{\bar{i}s\bar{j}} \mu_j \lambda,$$

so that $\beta_{jis} \mu_s = \beta_{isj} \mu_j$. Since $\beta_{isj} \leq \lambda$, by Lemma 2.2, we have $\beta_{jis} \mu_s \leq \lambda \mu_j$. Then (2.11) and (2.10) yield

$$\mu \mu_j \lambda \leq \mu^2 + \sum_{s \neq i, 1} \beta_{jis} \lambda \mu_j = \mu^2 + (\lambda - \mu) \lambda \mu_j.$$

Thus,

$$\mu \lambda \leq \frac{\mu^2}{\mu_j} + (\lambda - \mu) \lambda, \quad \text{for all } j \in S. \quad (2.12)$$

Let $\kappa \geq 1$ be the maximum value of μ_j , for all $j \in S$. Then by (2.12), $\mu \lambda \leq (\mu^2/\kappa) + \lambda^2 - \mu \lambda$, and hence $\mu^2 - 2\kappa\mu\lambda + \kappa\lambda^2 \geq 0$. So either $\mu \geq \kappa\lambda + \lambda\sqrt{\kappa^2 - \kappa}$ or $\mu \leq \kappa\lambda - \lambda\sqrt{\kappa^2 - \kappa}$. But the former contradicts $\mu < \lambda$. Therefore,

$$\mu \leq \lambda(\kappa - \sqrt{\kappa^2 - \kappa}), \quad \kappa = \max_{j \in S} \mu_j. \quad (2.13)$$

Suppose that $\kappa \leq 2$. Set $S_1 = \{j \in S \mid \mu_j = 1\}$, $S_2 = \{j \in S \mid \mu_j = 2\}$. Then $S = S_1 \dot{\cup} S_2$, and by (2.8), $\lambda = (|S_1| + |S_2|/2)\mu$. Then $\lambda > \mu > \lambda/2$ yields

$$2 > |S_1| + \frac{|S_2|}{2} > 1.$$

So $\kappa = 2$, and either $|S_2| = 3$ and $|S_1| = 0$, or $|S_2| = |S_1| = 1$. Hence, $\mu = 2\lambda/3$, which contradicts (2.13). Thus, $\kappa \geq 3$, and now (2.13) implies that $\mu \leq \lambda(3 - \sqrt{6})$. The inequality is strict, since μ and λ are integers, and the theorem is proved.

Remark 2.14. The proof of Theorem 2 also yields the following. If we assume, toward a contradiction, only that $\mu > \lambda/2$, then the entire argument through (2.12) remains valid. We may sum the inequality of (2.12) over $j \in S$, and use (2.8) to obtain $|S| \mu \lambda \leq \mu \lambda + |S|(\lambda - \mu)\lambda$, and therefore

$$\mu \leq (|S|/(2|S| - 1)) \cdot \lambda.$$

3. EXAMPLES

Several families of homogeneous table algebras, most of them integral, are described in detail in this section. Our first two examples illustrate some possibilities for the integers μ_i , where for (A, \mathbf{B}) a homogeneous ITA of degree λ and $b_i \in \mathbf{B}$, $(b_i, b_i) = \mu_i \lambda$ as in Proposition 1.11 and

Theorem 2. Example 3.1 shows that $\mu_i = \lambda/2$ can occur in a basis which also contains standard elements. Example 3.2 reveals that μ_i is not always a divisor of λ .

EXAMPLE 3.1. Fix any odd prime p . The cyclic group $\mathbb{Z}_{p-1} = \langle \phi \rangle$ acts fixed-point-freely on \mathbb{Z}_p , and acts (nonfaithfully if $p > 3$) by inversion on $\mathbb{Z}_3 = \{1, x, x^{-1}\}$. List $\mathbb{Z}_p \setminus \{1\}$ as $\{a_1, a_2, \dots, a_{p-1}\}$, where $a_{i+1} = \phi(a_i)$. Then $\langle \phi \rangle$ acts on $\mathbb{Z}_{p-1} \times \mathbb{Z}_3$ with the orbits

$$\begin{aligned} C_1 &:= \{a_1x, a_2x^{-1}, a_3x, a_4x^{-1}, \dots, a_{p-2}x, a_{p-1}x^{-1}\}, \\ C_2 &:= \{a_1x^{-1}, a_2x, a_3x^{-1}, a_4x, \dots, a_{p-2}x^{-1}, a_{p-1}x\}, \\ D &:= \{a_1, a_2, \dots, a_{p-2}, a_{p-1}\}, \quad E := \{x, x^{-1}\}, \quad \{1\}. \end{aligned}$$

Let G be the semidirect product $\mathbb{Z}_{p-1} \ltimes (\mathbb{Z}_p \times \mathbb{Z}_3)$. Then $C_1, C_2, D, E, \{1\}$ are the G -conjugacy classes which are contained in $\mathbb{Z}_p \times \mathbb{Z}_3$. Let $\mathbf{B} := \{1, \hat{C}_1, \hat{C}_2, \hat{D}, \hat{E}\} \subseteq \text{Cla}(G)$, $A := \langle \mathbf{B} \rangle \subseteq Z(\mathbb{C}G)$. Let $\lambda = p - 1$. Note that $a_{(p+1)/2} = a_1^{-1}$, so that $C_2 = C_1^{-1}$ iff $p \equiv 1 \pmod{4}$, and $C_i = C_i^{-1}$ ($i = 1, 2$) iff $p \equiv 3 \pmod{4}$. Now for $1 \leq i, j \leq 2$,

$$\begin{aligned} \hat{C}_i \hat{C}_j &= \begin{cases} \lambda 1 + (\text{combination of } \hat{C}_1, \hat{C}_2, \hat{D}), & \text{or} \\ (\lambda/2) \hat{E} + (\text{combination of } \hat{C}_1, \hat{C}_2, \hat{D}), \end{cases} \\ \hat{C}_i \hat{D} &= (\lambda/2) \hat{E} + (\text{combination of } \hat{C}_1, \hat{C}_2). \end{aligned}$$

Let $\mathbf{B}' = \{1, (\lambda/2)\hat{E}, \hat{C}_1, \hat{C}_2, \hat{D}\}$. It follows that (A, \mathbf{B}') is an ITA which is homogeneous of degree λ . Note that $\hat{C}_1, \hat{C}_2, \hat{D}$ are standard, and

$$((\lambda/2)\hat{E}, (\lambda/2)\hat{E}) = (\lambda^2/4)(\hat{E}, \hat{E}) = (\lambda^2/4) \cdot 2 = (\lambda/2)\lambda,$$

so that $\mu_{(\lambda/2)\hat{E}} = \lambda/2$.

EXAMPLE 3.2. This example is inspired by the P -polynomial (standard) C -algebras defined in [BI, Sect. III.6], although we do not explicitly use that construction here. Fix $\lambda, \mu, \beta \in \mathbb{R}^+$ with $\lambda \geq \mu\beta$, $\lambda \geq \beta + 1$, and $\lambda + \mu^2\beta \geq \mu(\lambda + 1)$. We denote a basis $\mathbf{V} = \mathbf{V}(3, \lambda, \mu, \beta)$ for a 3-dimensional \mathbb{C} -vector space A as $\mathbf{V} := \{1, b, c\}$. We define multiplication so that 1 is the multiplicative identity, and so that

$$\begin{aligned} b^2 &= \lambda 1 + (\lambda - \beta - 1)b + \beta c, \\ bc &= cb = \mu\beta b + (\lambda - \mu\beta)c, \\ c^2 &= \mu\lambda 1 + \mu(\lambda - \mu\beta)b + (\lambda + \mu^2\beta - \mu - \mu\lambda)c. \end{aligned} \tag{3.2a}$$

This product extends bilinearly to a multiplication on A which is commutative and also associative, since

$$\begin{aligned} b(bc) &= \mu\beta\lambda 1 + (\mu\beta(2\lambda - \beta - \mu\beta - 1))b + (\mu\beta^2 + (\lambda - \mu\beta)^2)c \\ &= b^2c, \end{aligned}$$

and

$$\begin{aligned} c(cb) &= (\lambda - \mu\beta)\mu\lambda 1 + (\mu^2\beta^2 + \mu(\lambda - \mu\beta)^2)b \\ &\quad + ((\lambda - \mu\beta)(\mu\beta + \lambda + \mu^2\beta - \mu - \mu\lambda))c = c^2b. \end{aligned}$$

Then (A, \mathbf{V}) satisfies Definition 1.1, with $\bar{} = \text{id}_A$. Substitution of λ for each of b and c in (3.2a) yields equalities. Hence, $f(b) = \lambda = f(c)$ and (A, \mathbf{V}) is homogeneous of degree λ . Note that $(b, b) = \lambda$, so that b is standard, and $(c, c) = \mu\lambda$.

If $\lambda, \mu, \beta \in \mathbb{Z}^+$ then (A, \mathbf{V}) is clearly integral. Note that if we select $0 \leq \gamma < \lambda$ and choose μ, β with $\mu\beta = \lambda - \gamma$, then the condition $\lambda + \mu^2\beta \geq \mu(\lambda + 1)$ is equivalent to $\mu \leq \lambda/(1 + \gamma)$. Hence, given $\lambda \in \mathbb{Z}^+$, there are usually many choices for μ, β so that $(A, \mathbf{V}(3, \lambda, \mu, \beta))$ is a homogeneous ITA of degree λ , and with $\mu \nmid \lambda$.

Our next example presents the family $\mathbf{T}(n, \lambda, m)$ of certain homogeneous table algebras of degree λ . When λ is an odd integer at least 3, the algebras will be integral.

EXAMPLE 3.3. Fix $n, m \in \mathbb{Z}$ with $n \geq 0, m \geq 1$. Fix $\lambda \in \mathbb{R}$ with $\lambda > 1$. Let $\alpha = (\lambda - 1)/2, \beta = (\lambda + 1)/2$ (so $\alpha > 0$ since $\lambda > 1$).

Let $\mathbb{Z}_m = \langle u \rangle$, the cyclic group of order m . Let $R = \mathbb{C}\langle u \rangle$ be the group algebra with automorphism ϕ such that $\phi(u) = \bar{u} := u^{-1}$. (So $\phi(\sum_{i=0}^{m-1} \lambda_i u^i) = \sum_{i=0}^{m-1} \lambda_i \bar{u}^i, \lambda_i \in \mathbb{C}$.) Let b be an indeterminate over R , so that $R[b]$ is a polynomial ring. We proceed to define recursively the following elements of $R[b]$:

$$b_0 := b,$$

and for all $i > 0$,

$$b_i := \beta^{-1}(bb_{i-1} - \alpha b_{i-1}). \quad (3.3a)$$

Thus, each b_i has degree $i + 1$ and leading coefficient β^{-i} , a unit in R . Also,

$$bb_j = \alpha b_j + \beta b_{j+1}, \quad \text{all } j \geq 0. \quad (3.3b)$$

It follows from (3.3a), (3.3b), and induction on i that

$$b_i b_j = \alpha b_{i+j} + \beta b_{i+j+1}, \quad \text{all } i, j \geq 0. \quad (3.3c)$$

Now let $p(b) := \beta b_{n+1} - \lambda u1 - \alpha ub$. Then $p(b)$ has degree $n + 2$ and leading coefficient β^{-n} , a unit in R . Let $\mathcal{A} := R[b]/\langle p(b) \rangle$, which has $\{1, b_0, b_1, \dots, b_n\}$ as an R -basis, and hence has the \mathbb{C} -basis

$$\mathbf{B} := \{1\} \dot{\cup} \{\lambda u^j | 0 < j < m\} \dot{\cup} \{b_i u^j | 0 \leq i \leq n, 0 \leq j < m\}.$$

We identify these elements of $R[b]$ with their images in \mathcal{A} . By definition of $p(b)$,

$$\beta b_{n+1} \equiv \lambda u1 + \alpha ub_0 \pmod{\langle p(b) \rangle}.$$

So for $i + j = n$,

$$b_i b_j = \alpha b_n + \beta b_{n+1} \equiv \lambda u1 + \alpha ub_0 + \alpha b_n \pmod{\langle p(b) \rangle}. \quad (3.3d)$$

Now

$$\begin{aligned} \beta^2 b_{n+2} &= \beta b b_{n+1} - \beta \alpha b_{n+1} \equiv b(\lambda u1 + \alpha ub_0) - \beta \alpha b_{n+1} \\ &\pmod{\langle p(b) \rangle}. \end{aligned} \quad (3.3e)$$

Note that

$$b(\lambda u1 + \alpha ub_0) = \lambda ub_0 + \alpha^2 ub_0 + \alpha \beta ub_1 = \beta^2 ub_0 + \alpha \beta ub_1,$$

since $\alpha^2 + \lambda = \beta^2$. So multiplication of (3.3e) by β^{-1} yields

$$\beta b_{n+2} + \alpha b_{n+1} \equiv \alpha ub_1 + \beta ub_0 \pmod{\langle p(b) \rangle}.$$

One can now verify that for all $i > 0$,

$$\beta b_{n+i+1} + \alpha b_{n+i} \equiv \alpha ub_i + \beta ub_{i-1} \pmod{\langle p(b) \rangle}, \quad (3.3f)$$

simply by using induction on i , assuming (3.3f) true for some i , multiplying the congruence through by b , and applying (3.3b) and (3.3f) for i to obtain (3.3f) for $i + 1$. Now (3.3c), (3.3d), (3.3f) imply the following formulas for products in \mathcal{A} . For all $0 \leq i, j \leq n$,

$$b_i b_j = \begin{cases} \alpha b_{i+j} + \beta b_{i+j+1}, & i + j < n \\ \lambda u1 + \alpha ub_0 + \alpha b_n, & i + j = n \\ \alpha ub_{i+j-n} + \beta ub_{i+j-n-1}, & i + j > n. \end{cases} \quad (3.3g)$$

The remaining products of pairs of elements of \mathbf{B} are obtained by multiplying (3.3g) through by powers of u .

Define the map $\psi: A \rightarrow A$ so that $\psi(b_i) = \bar{u}b_{n-i}$, $0 \leq i \leq n$, $\psi(1) = 1$, $\psi(rx) = \phi(r)\psi(x)$ for all $x = b_i$ or 1 and all $r \in R$, and so that ψ preserves addition on A . Then one can check from (3.3g) that $\psi(b_i b_j) = \psi(b_i)\psi(b_j)$ for all $0 \leq i, j \leq n$. It follows that ψ is a \mathbb{C} -algebra automorphism of A , of order 2 (unless $n = 0$ and $m = 1$), and with $\psi(\mathbf{B}) = \mathbf{B}$.

Now by (3.3g), (A, \mathbf{B}) is a table algebra with respect to the automorphism ψ . Substitution of λ for each b_i , and of 1 for u , in (3.3g) yields equalities. Hence, (A, \mathbf{B}) is homogeneous of degree λ . Define $\mathbf{T}(n, \lambda, m) := \mathbf{B}$. Note that each $b_i \in \mathbf{B}$ is a faithful element. If λ is an odd integer (≥ 3), then $\alpha, \beta \in \mathbb{Z}^+$, and so (3.3g) implies that $\mathbf{T}(n, \lambda, m)$ is integral.

Clearly $\mathbf{L}(\mathbf{B}) = \{1\} \cup \{\lambda u^j | 0 < j < m\}$, and $\mathbf{B}/\mathbf{L}(\mathbf{B}) \cong_x \mathbf{T}(n, \lambda, 1)$.

DEFINITION 3.4. For all $n \in \mathbb{Z}^+ \cup \{0\}$ and $\lambda \in \mathbb{R}$ with $\lambda > 1$,

$$\mathbf{T}_n(\lambda) := \mathbf{T}(n, \lambda, 1).$$

We summarize our results in

PROPOSITION 3.5. For all $n \in \mathbb{Z}^+ \cup \{0\}$, $m \in \mathbb{Z}^+$, and $\lambda \in \mathbb{R}$ with $\lambda > 1$, $(A, \mathbf{T}(n, \lambda, m))$ is a homogeneous table algebra of degree λ , and is integral when λ is an odd integer. Also,

$$\mathbf{T}(n, \lambda, m)/\mathbf{L}(\mathbf{T}(n, \lambda, m)) \cong_x \mathbf{T}_n(\lambda).$$

Finally, $\mathbf{T}(n, \lambda, m)$ for $n > 1$ is not isomorphic to a table subset of $(Z(\mathbb{C}G), \text{Cla}(G))$ or $(\text{Ch}(G), \text{Irr}(G))$ for any finite group G .

Proof. The first two statements are already verified. For the final claim, note that when $n > 1$, $b_0, \bar{b}_0 = \psi(b_0) = \bar{u}b_n$ and $\bar{u}b_{n-1}$ are distinct elements of $\mathbf{T}(n, \lambda, m)$, for any $\lambda > 1$ and $m \in \mathbb{Z}^+$, with

$$b_0(\bar{u}b_{n-1}) = \alpha \bar{u}b_{n-1} + \beta \bar{u}b_n = \alpha \bar{u}b_{n-1} + \beta \bar{b}_0,$$

by (3.3g). However, for any finite group G , neither $\text{Cla}(G)$ nor $\text{Irr}(G)$ contains distinct elements of the form b, \bar{b}, c such that $\text{Supp}(bc) = \{c, \bar{b}\}$ [AB, Corollary E].

Remark 3.6. The last statement of Proposition 3.5 is false for $n = 0, 1$. Note that for any odd prime p , $\mathbf{T}_0(p-1) \cong_x \mathbf{O}(p-1, \mathbb{Z}_p)$; and if $p \equiv 3 \pmod{4}$ then $\mathbf{T}_1((p-1)/2) \cong_x \mathbf{O}((p-1)/2, \mathbb{Z}_p)$.

The final result in this section verifies the existence of $\mathbf{N}((3), \lambda)$, as defined in Example 1.26.

PROPOSITION 3.7. Fix $\lambda \in \mathbb{Z}^+$. Let A be a complex vector space with a basis $\mathbf{C} = \{1, \lambda u, \lambda u^2, \dots, \lambda u^{\lambda-1}, v_1, v_2, \dots, v_{\lambda-1}\}$ such that $\langle u \rangle = \mathbb{Z}_\lambda$ as a multiplicative group. Extend this multiplication to a bilinear product on A such that $v_i v_j = \lambda v_{i+j}$ ($i + j$ read mod λ) if $i + j \neq \lambda$, $v_j v_{\lambda-j} = \lambda \cdot \sum_{i=0}^{\lambda-1} u^i$, and $v_j u^i = u^i v_j = v_j$ for all i, j . Then the following hold:

(i) The given multiplication is commutative and associative on A , and (A, \mathbf{C}) is a homogeneous ITA of degree λ , where $\bar{u}^i = u^{-i}$ and $\bar{v}_i = v_{\lambda-i}$ for all i .

(ii) Let b be an indeterminate over the commutative ring A , so that $A[b]$ is a polynomial ring. Let I be the \mathbb{C} -span of all $v_i - \sum_{j=0}^{\lambda-1} u^j$, $1 \leq i \leq \lambda - 1$, a subspace of dimension $\lambda - 1$. Then I is an ideal of A . Furthermore, if J is the ideal of $A[b]$ generated by Ib and $b^2 - (\lambda \cdot 1 + \sum_{i=1}^{\lambda-1} v_i)$, and if \mathbf{B} is the set of images in $A[b]/J$ of \mathbf{C} and $\{bu^i | 0 \leq i \leq \lambda\}$, then $(A[b]/J, \mathbf{B})$ is the algebra $\mathbf{N}((3), \lambda)$ of Example 1.26.

Proof. Part (i) is easy to check directly. For (ii), note first that for $1 \leq i, t, r \leq \lambda - 1$,

$$v_t \left(v_i - \sum_{j=0}^{\lambda-1} u^j \right) = v_t v_i - \lambda v_t = \begin{cases} \lambda v_{t+i} - \lambda v_t & \text{if } i + t \neq \lambda, \\ -\lambda \left(v_t - \sum_{j=0}^{\lambda-1} u^j \right) & \text{if } i + t = \lambda; \end{cases}$$

$$= \begin{cases} \lambda \left(v_{t+i} - \sum_{j=0}^{\lambda-1} u^j - \left(v_t - \sum_{j=0}^{\lambda-1} u^j \right) \right) \in I \\ -\lambda \left(v_t - \sum_{j=0}^{\lambda-1} u^j \right) \in I; \end{cases}$$

$$u^r \left(v_i - \sum_{j=0}^{\lambda-1} u^j \right) = v_i - \sum_{j=0}^{\lambda-1} u^j \in I.$$

So I is an ideal, and $R := A/I$ has as a \mathbb{C} -basis (the images of) $1, \lambda u, \lambda u^2, \dots, \lambda u^{\lambda-1}$. Then as an algebra,

$$A[b]/Ib \cong A \oplus Rb \oplus Rb^2 \oplus \dots,$$

where each Rb^n has as a \mathbb{C} -basis $\{b^n, ub^n, u^2 b^n, \dots, u^{\lambda-1} b^n\}$. Now J/Ib is the ideal of $A[b]/Ib$ generated by (the image of) $b^2 - (\lambda \cdot 1 + \sum_{i=1}^{\lambda-1} v_i)$. Hence, $A[b]/J \cong A \oplus Rb$, which has \mathbb{C} -basis (the images of) \mathbf{C} and bu^i , for $0 \leq i \leq \lambda - 1$. In this algebra, $b^2 = \lambda \cdot 1 + \sum_{i=1}^{\lambda-1} v_i$, and $b(v_i - \sum_{j=0}^{\lambda-1} u^j) = 0$ implies that $bv_i = \sum_{j=0}^{\lambda-1} bu^j$. Therefore, \mathbf{B} spans a table algebra which is exactly isomorphic to $\mathbf{N}((3), \lambda)$.

4. PROOF OF THEOREM 3

Throughout this section, (A, \mathbf{B}) is a homogeneous ITA of degree λ , $\lambda > 2$, and $b \in \mathbf{B}$ is a standard, faithful element of width 2. If $b = \bar{b}$, then [X, Theorem 1] shows that $\mathbf{B} \cong_x \mathbf{T}_0(\lambda)$, $(\mathbf{T}_0(\lambda) \otimes \mathbb{Z}_2)'$, $\mathbf{V}(3, \lambda)$, or $(\mathbf{V}(3, \lambda) \otimes \mathbb{Z}_2)'$. So we may assume henceforth that $b \neq \bar{b}$.

By hypothesis, $\text{Supp}(b\bar{b}) = \{1, c\}$ for some $c \in \mathbf{B} \setminus \{1\}$. Then $c = \bar{c}$, as $b\bar{b}$ is invariant under $\bar{}$. Since b is standard, 1 appears with coefficient λ in the decomposition of $b\bar{b}$ in terms of \mathbf{B} . Then $\text{degree } f(c) = \lambda$, while $f(b\bar{b}) = \lambda^2$, implies that

$$b\bar{b} = \lambda 1 + (\lambda - 1)c. \quad (4.1)$$

Now $(c, c) = \mu\lambda$ for some $\mu \in \mathbb{Z}$, $1 \leq \mu \leq \lambda$, by Proposition 1.11. Let γ be the coefficient of b in the decomposition of bc . Then by (1.7),

$$\lambda\gamma = (b, b)\gamma = (bc, b) = (b\bar{b}, c) = (\lambda - 1)(c, c) = (\lambda - 1)\mu\lambda.$$

So $\gamma = (\lambda - 1)\mu$. But by Lemma 2.2, $\gamma \leq \lambda$. This forces $\mu = 1$, as $\lambda > 2$, and so $\gamma = \lambda - 1$. Therefore,

c is standard,

and

$$bc = (\lambda - 1)b + d, \quad \text{some } d \in \mathbf{B} \setminus \{1, b\}. \quad (4.2)$$

Proposition 1.11, (4.2), and (1.7) yield that there exists $\delta \in \mathbb{Z}$ with $1 \leq \delta \leq \lambda$ such that

$$\delta\lambda = (d, d) = (bc, d) = (c, \bar{b}d).$$

As $(c, c) = \lambda$, c appears with coefficient δ in $\bar{b}d$. By (4.1) and (4.2),

$$\begin{aligned} \bar{b}(bc) &= (\lambda - 1)\bar{b}b + \bar{b}d = (\lambda - 1)\lambda 1 + (\lambda - 1)^2 c + \bar{b}d \\ &= (\bar{b}b)c = \lambda c + (\lambda - 1)c^2. \end{aligned}$$

Hence,

$$(\lambda - 1)c^2 = (\lambda - 1)\lambda 1 + ((\lambda - 1)^2 - \lambda)c + \bar{b}d. \quad (4.3)$$

Now c appears with net coefficient $(\lambda - 1)^2 - \lambda + \delta$ on the right hand side of (4.3), so (4.3) implies that $\lambda - 1 \mid \lambda - \delta$. Thus $\delta = 1$ or λ . That is, as noted in Remark 1.12,

$$d \text{ is either standard or linear.} \quad (4.4)$$

Suppose that d is linear. Then $c \in \text{Supp}(\bar{b}d)$ implies that $b \in \text{Supp}(dc)$, and hence $dc = \lambda b$ and $\bar{b}d = \lambda c$ [AB, Proposition 3.2]. Then (4.3) yields $c^2 = \lambda 1 + (\lambda - 1)c$, and hence $\mathbf{B}_c = \{1, c\} \cong_x \mathbf{T}_0(\lambda)$. Also, $\mathbf{B}_d \cong_x \mathbb{Z}'_m$ for some $m > 0$. Since $\mathbf{B}_c \cap \mathbf{B}_d = \{1\}$, and the members of \mathbf{B}_d are linear, it follows that the elements 1 , $\lambda^{-i+1}d^i$ for $1 \leq i < m$, and $\lambda^{-i}cd^i$ for $0 \leq i < m$, are all distinct. It is easily seen that they form a table subset \mathbf{C} which is exactly isomorphic to $(\mathbf{B}_c \otimes \mathbb{Z}'_m)' \cong_x (\mathbf{T}_0(\lambda) \otimes \mathbb{Z}'_m)'$. Since $b = \lambda^{-1}cd \in \mathbf{C}$ and b is faithful, $\mathbf{B} = \mathbf{C} \cong_x (\mathbf{T}_0(\lambda) \otimes \mathbf{Z}'_m)'$.

Suppose that d is standard. Then c has coefficient $\delta = 1$ in $\bar{b}d$. By (4.3), the coefficient of any other element of $\text{Supp}(\bar{b}d)$ is divisible by $\lambda - 1$. Thus, homogeneity and $d \neq b$ yield

$$\bar{b}d = c + (\lambda - 1)g, \quad \text{some } g \in \mathbf{B} \setminus \{1, c\}. \quad (4.5)$$

Substitution of (4.5) into (4.3) implies that

$$c^2 = \lambda 1 + (\lambda - 2)c + g. \quad (4.6)$$

Now by (4.5) and (4.2),

$$\begin{aligned} c(\bar{b}d) &= c^2 + (\lambda - 1)cg = \lambda 1 + (\lambda - 2)c + g + (\lambda - 1)cg \\ &= \bar{b}cd = ((\lambda - 1)\bar{b} + \bar{d})d = (\lambda - 1)\bar{b}d + \bar{d}d \\ &= (\lambda - 1)c + (\lambda - 1)^2g + \bar{d}d. \end{aligned}$$

It follows that g must occur with coefficient at least $(\lambda - 1)^2$ in $g + (\lambda - 1)cg$, and hence with coefficient at least $\lambda - 1$ in cg (as $\lambda > 2$). Since $g \in \text{Supp}(c^2)$ forces $c \in \text{Supp}(cg)$, we must have

$$cg = c + (\lambda - 1)g. \quad (4.7)$$

It follows from (4.6) and (4.7) that $\mathbf{B}_c = \{1, g, c\} \cong_x \mathbf{V}(3, \lambda)$, where $g \leftrightarrow b$ in Example 1.17. In particular, $g^2 = \lambda 1 + (\lambda - 1)c$.

Now $(bg, d) = (g, \bar{b}d) = (\lambda - 1)(g, g) = (\lambda - 1)\lambda$. Since $(d, d) = \lambda$, we have

$$bg = (\lambda - 1)d + u, \quad \text{some } u \in \mathbf{B} \setminus \{1, d\}. \quad (4.8)$$

Then by (1.7),

$$\begin{aligned} (u, u) + (\lambda - 1)^2\lambda &= (bg, bg) = (b\bar{b}, g^2) \\ &= (\lambda 1 + (\lambda - 1)c, \lambda 1 + (\lambda - 1)c) = \lambda^2 + (\lambda - 1)^2\lambda. \end{aligned}$$

Hence, $(u, u) = \lambda^2$ and u is linear. Since $(b, gu) = (bg, u) > 0$, we have by [AB, Proposition 3.2] that $gu = \lambda b$. Now $\mathbf{B}_u \cong_x \mathbb{Z}'_m$ for some $m > 0$. Since $\mathbf{B}_c \cap \mathbf{B}_u = \{1\}$, and the members of \mathbf{B}_u are linear, it follows that the elements $1, \lambda^{-i+1}u^i$ for $1 \leq i < m$, $\lambda^{-i}cu^i$ for $0 \leq i < m$, and $\lambda^{-i}gu^i$ for $0 \leq i < m$ are all distinct. They clearly form a table subset \mathbf{D} , with $\mathbf{D} \cong_x (\mathbf{V}(3, \lambda) \otimes \mathbb{Z}_m)'$. Since faithful $b = \lambda^{-1}gu \in \mathbf{D}$, $\mathbf{B} = \mathbf{D}$. The proof of Theorem 3 is complete.

5. NILPOTENT HOMOGENEOUS ITAs

Theorems 4 and 5, which concern nilpotent homogeneous ITAs, are proved in this section. But first we establish some lemmas of a more general nature.

LEMMA 5.1. *Let (A, \mathbf{B}) be a table algebra, $\mathbf{L} = \mathbf{L}(\mathbf{B})$, and $b \in \mathbf{B}$. Then $\text{Supp}_{\mathbf{B}}(b\bar{b}) \cap \mathbf{L} = \text{sta}_{\mathbf{L}}(b)$.*

Proof. For any $c \in \mathbf{B}$, (1.9) implies that $c \in \text{Supp}_{\mathbf{B}}(b\bar{b}) \Leftrightarrow b \in \text{Supp}_{\mathbf{B}}(bc)$. If $c \in \mathbf{L}$, then [AB, Proposition 3.2] yields that $b \in \text{Supp}_{\mathbf{B}}(bc) \Leftrightarrow \{b\} = \text{Supp}_{\mathbf{B}}(bc) \Leftrightarrow c \in \text{sta}_{\mathbf{L}}(b)$. The result follows.

LEMMA 5.2. *Assume that (A, \mathbf{B}) is a homogeneous ITA of degree λ . Let $\mathbf{L} = \mathbf{L}(\mathbf{B})$, $e = e_{\mathbf{L}}$, and $b \in \mathbf{B}$. Let b have length $\mu_b \lambda$ (as in Proposition 1.11), and b have length $\mu_{be} \lambda$, with respect to the algebra \mathbf{B}/\mathbf{L} . Then $\mu_{be} = \mu_b \cdot |\text{sta}_{\mathbf{L}} b|$.*

Proof. By definition of length and Lemma 5.1,

$$b\bar{b} = \lambda \mu_b \cdot 1 + \sum_{b_i \in \text{sta}_{\mathbf{L} \setminus \{1\}} b} \gamma_i b_i + \sum_{b_i \in \text{Supp}_{\mathbf{B} \setminus \mathbf{L}}(b\bar{b})} \gamma_i b_i,$$

where each $\gamma_i \in \mathbb{Z}^+$. But for all $b_i \in \text{sta}_{\mathbf{L} \setminus \{1\}} b$, $b_i b = \lambda b$ and (1.7) yield

$$\gamma_i \lambda^2 = \gamma_i (b_i, b_i) = (b\bar{b}, b_i) = (b, bb_i) = (b, \lambda b) = \lambda \cdot \mu_b \lambda.$$

Thus, $\gamma_i = \mu_b$ and

$$b\bar{b} = \mu_b \left(\lambda 1 + \sum_{b_i \in \text{sta}_{\mathbf{L} \setminus \{1\}} b} b_i \right) + \sum_{b_i \in \text{Supp}_{\mathbf{B} \setminus \mathbf{L}}(b\bar{b})} \gamma_i b_i. \quad (5.3)$$

Since $b_i \in \mathbf{L} \setminus \{1\}$ implies that $b_i e = \lambda e$, we have

$$\begin{aligned} be\bar{b}e &= \mu_b \left(\lambda e + \sum_{b_i \in \text{sta}_{\mathbf{L} \setminus \{1\}} b} b_i e \right) + \sum_{b_i \in \text{Supp}_{\mathbf{B} \setminus \mathbf{L}}(b\bar{b})} \gamma_i b_i e \\ &= \mu_b \cdot |\text{sta}_{\mathbf{L}} b| \cdot \lambda e + (\text{combination of } (\mathbf{B}/\mathbf{L}) \setminus (\mathbf{L}/\mathbf{L})). \end{aligned}$$

By Proposition 1.11 and the definition of length, $\mu_{be} \lambda$ is the coefficient of e in the \mathbf{B}/\mathbf{L} decomposition of $be\bar{b}e$. Hence, $\mu_{be} = \mu_b \cdot |\text{sta}_{\mathbf{L}} b|$.

LEMMA 5.4. *Assume that (A, \mathbf{B}) is a homogeneous ITA of degree λ . Suppose that $b, c \in \mathbf{B} \setminus \mathbf{L}$ are such that $b \neq \bar{c}$ and bc is the sum of λ distinct standard elements of \mathbf{B} . Then b and c are standard and $\text{Supp}_{\mathbf{B}}(b\bar{b}) \cap \text{Supp}_{\mathbf{B}}(c\bar{c}) = \{1\}$. In particular, $b \neq c$.*

Proof. Set $bc = \sum_{i=1}^{\lambda} b_i$, where each b_i is standard, and there are λ distinct such elements in the sum. Then $(bc, bc) = \sum_{i=1}^{\lambda} (b_i, b_i) = \lambda^2$. By (1.7), $\lambda^2 = (b\bar{b}, c\bar{c})$. But by Proposition 1.11, $b\bar{b} = \mu_b \lambda \cdot 1 + x$ and $c\bar{c} = \mu_c \lambda \cdot 1 + y$, where $\mu_b, \mu_c \in \mathbb{Z}^+$ and x, y are \mathbb{Z}^+ -combinations of nonempty subsets of $\mathbf{B} \setminus \{1\}$ (as b, c are nonlinear). Thus,

$$\lambda^2 = (b\bar{b}, c\bar{c}) = \mu_b \mu_c \lambda^2 + (x, y).$$

Hence, $\mu_b = \mu_c = 1$ and $(x, y) = 0$. The result follows.

Proof of Theorem 4. We use induction on n . Let $\mathbf{L}^{(j)}(\mathbf{B}) = \mathbf{L}^{(j)}$, all $j \geq 0$. If $n = 1$ then $\mathbf{B} = \mathbf{L}$, $\mu = \lambda$, and the theorem holds. Suppose that $n > 1$. Then \mathbf{B}/\mathbf{L} has nilpotence class $n - 1$, $be_{\mathbf{L}} \in \mathbf{B}/\mathbf{L}$ (unless $b \in \mathbf{L}$, in which case $\mu = \lambda$ and the conclusion holds), and $\mathbf{L}^{(j)}(\mathbf{B}/\mathbf{L}) = \mathbf{L}^{(j+1)}/\mathbf{L}$ for all $j \geq 0$. By induction,

$$\lambda = \mu_{be_{\mathbf{L}}} \cdot |\text{sta}_{\mathbf{L}^{(2)}/\mathbf{L}}(be_{\mathbf{L}})| \cdot |\text{sta}_{\mathbf{L}^{(3)}/\mathbf{L}^{(2)}}(be_{\mathbf{L}^{(2)}})| \cdots |\text{sta}_{\mathbf{L}^{(n)}/\mathbf{L}^{(n-1)}}(be_{\mathbf{L}^{(n-1)}})|.$$

Lemma 5.2 implies that $\mu_{be_{\mathbf{L}}} = \mu \cdot |\text{sta}_{\mathbf{L}} b|$, and the theorem is proved.

Proof of Theorem 5. Let $\mathbf{L}^{(j)}(\mathbf{B}) = \mathbf{L}^{(j)}$ for all $j \geq 0$. Since \mathbf{B} is nilpotent and λ is a prime, it follows from Theorem 4 and Remark 1.12 that every element of $\mathbf{B} \setminus \mathbf{L}$ is standard. Let \mathbf{B} have nilpotence class n .

For any table subset \mathbf{M} of \mathbf{L} , let $(1/\lambda)\mathbf{M}$ denote the set

$$(1/\lambda)\mathbf{M} := \{1\} \cup \{(1/\lambda)x \mid x \in \mathbf{M} \setminus \{1\}\}.$$

Thus, $(1/\lambda)\mathbf{M}$ forms an abelian group, every member of which has degree 1. Furthermore, for all $y \in \mathbf{B}$ and $u \in (1/\lambda)\mathbf{M}$, yu is in \mathbf{B} by [AB, Proposition 3.2].

Suppose that $n = 1$. Then b is linear, faithful, and real. So $b^2 = \lambda^2 1$ implies that $\mathbf{B} = \{1, b\}$ and $\mathbf{B} \cong_x \mathbb{Z}'_2$.

Suppose that $n = 2$. Then $b \in \mathbf{L}^{(2)} \setminus \mathbf{L}$. Since $\mathbf{L}^{(2)}/\mathbf{L}$ is abelian, by definition, $\text{Supp}_{\mathbf{B}}(b^2) = \text{Supp}_{\mathbf{B}}(b\bar{b}) \subseteq \mathbf{L}$. Then by Lemma 5.1 and (5.3), and since $\mu_b = 1$,

$$b^2 = \lambda 1 + \sum_{b_i \in \text{sta}_{\mathbf{L} \setminus \{1\}} b} b_i.$$

Thus, $\lambda^2 = f(b)^2 = \lambda \cdot |\text{sta}_{\mathbf{L}} b|$, so that $|\text{sta}_{\mathbf{L}} b| = \lambda$. Since $\text{Supp}_{\mathbf{B}}(b^2) = \text{sta}_{\mathbf{L}} b$, it follows that $\text{Supp}_{\mathbf{B}}(b^t)$ is $\{b\}$ for all odd t , and is $\text{sta}_{\mathbf{L}} b$ for all even t . As b is faithful, $\mathbf{B} = \{b\} \cup \text{sta}_{\mathbf{L}} b$. Since λ is a prime, $\text{sta}_{\mathbf{L}} b \cong_x \mathbb{Z}'_{\lambda}$. Therefore, $\mathbf{B} \cong_x \mathbf{N}((2), \lambda)$.

Suppose that $n = 3$. Then $b \in \mathbf{L}^{(3)} \setminus \mathbf{L}^{(2)}$. By the previous case, $b^2 e_{\mathbf{L}}$ equals $\lambda e_{\mathbf{L}}$ plus a sum of $\lambda - 1$ distinct, nontrivial elements of $\mathbf{L}^{(2)}/\mathbf{L}$. Hence,

$$b^2 = \lambda 1 + \sum_{i=1}^{\lambda-1} v_i, \quad (5.5)$$

where the v_i are distinct elements of $\mathbf{L}^{(2)} \setminus \mathbf{L}$ with distinct images in $\mathbf{L}^{(2)}/\mathbf{L}$. Since, by the previous case,

$$e_{\mathbf{L}} \cup \{v_i e_{\mathbf{L}}\}_{i=1}^{\lambda-1} = \text{sta}_{\mathbf{L}^{(2)}/\mathbf{L}}(be_{\mathbf{L}}) \cong_x \mathbb{Z}'_{\lambda},$$

we may index the v_i so that

$$v_i e_{\mathbf{L}} v_j e_{\mathbf{L}} = \begin{cases} \lambda v_{i+j} e_{\mathbf{L}} & \text{if } i+j \neq \lambda \text{ (read } i+j \text{ mod } \lambda), \\ \lambda^2 e_{\mathbf{L}} & \text{if } i+j = \lambda. \end{cases} \quad (5.6)$$

Since $b^2 = \overline{b^2}$, we have that $v_{\lambda-i} = \overline{v_i}$ for $1 \leq i \leq \lambda - 1$.

Fix i, j with $i+j \neq \lambda$. By (5.6), $\text{Supp}_{\mathbf{B}}(v_i v_j) \subseteq \text{Supp}_{\mathbf{B}}(v_{i+j} e_{\mathbf{L}})$. Since $(b, b\overline{v_i}) = (b^2, v_{\lambda-i}) > 0$ and $(b, bv_j) = (b^2, v_j) > 0$,

$$\lambda = (b, b) \leq (b\overline{v_i}, bv_j) = (b^2, v_i v_j).$$

As $\text{Supp}_{\mathbf{B}}(b^2) \cap \text{Supp}_{\mathbf{B}}(v_{i+j}e_{\mathbf{L}}) = \{v_{i+j}\}$, we may conclude that

$$v_{i+j} \in \text{Supp}_{\mathbf{B}}(v_i v_j), \quad \text{for all } i, j \text{ with } i + j \neq \lambda. \quad (5.7)$$

By Lemma 5.1 and (5.3), for all $1 \leq i \leq \lambda - 1$,

$$v_i \overline{v_i} = \lambda 1 + \sum_{w \in \text{sta}_{\mathbf{L} \setminus \{\mathbf{1}\}} v_i} w. \quad (5.8)$$

In particular, $\text{sta}_{\mathbf{L}} v_i \cong_x \mathbb{Z}'_{\lambda}$ for all i . Suppose that $\text{sta}_{\mathbf{L}} v_j \neq \text{sta}_{\mathbf{L}} v_1$ for some j . Then let $i \geq 1$ be minimal such that $\text{sta}_{\mathbf{L}} v_{i+1} \neq \text{sta}_{\mathbf{L}} v_1$ (so $\text{sta}_{\mathbf{L}} v_i = \text{sta}_{\mathbf{L}} v_1$). Then $\text{sta}_{\mathbf{L}} v_{i+1} \cap \text{sta}_{\mathbf{L}} v_1 = \{1\}$ and $\{v_{i+1} w' \mid w' \in (1/\lambda) \text{sta}_{\mathbf{L}} v_1\}$ is a set of λ distinct elements of $\mathbf{L}^{(2)} \setminus \mathbf{L}$. Since $w' v_1 v_i = v_1 v_i$ for all $w' \in (1/\lambda) \text{sta}_{\mathbf{L}} v_1$, it follows from (5.7) that

$$v_1 v_i = \sum_{w' \in (1/\lambda) \text{sta}_{\mathbf{L}} v_1} v_{i+1} w',$$

a sum of λ distinct standard elements of \mathbf{B} . As $v_1 \overline{v_1} = v_i \overline{v_i}$ by (5.8), this contradicts Lemma 5.4. We have established that

$$\text{sta}_{\mathbf{L}} v_i = \text{sta}_{\mathbf{L}} v_1, \quad 1 \leq i \leq \lambda - 1. \quad (5.9)$$

For all i, j with $j \neq \lambda - i$, (5.6) implies that $v_i v_j = \sum_t \gamma_t y_t$, where the y_t are distinct elements of $\text{Supp}_{\mathbf{B}}(v_{i+j}e_{\mathbf{L}})$, one of which is v_{i+j} by (5.7), each $\gamma_t \in \mathbb{Z}^+$, and $\sum_t \gamma_t = \lambda$. Thus,

$$\begin{aligned} \lambda \cdot \sum_t \gamma_t^2 &= \sum_t \gamma_t^2 (y_t, y_t) = (v_i v_j, v_i v_j) = (v_i \overline{v_i}, v_j \overline{v_j}) \\ &= \left(\lambda 1 + \sum_{\text{sta}_{\mathbf{L} \setminus \{\mathbf{1}\}} v_1} w, \lambda 1 + \sum_{\text{sta}_{\mathbf{L} \setminus \{\mathbf{1}\}} v_1} w \right) = \lambda \cdot \lambda^2 = \lambda^3. \end{aligned}$$

Hence, $\sum_t \gamma_t^2 = \lambda^2 = (\sum_t \gamma_t)^2$. It follows that there is only one such t , so that

$$v_i v_j = \lambda v_{i+j}, \quad \text{for all } i, j \text{ with } i \neq \lambda - j. \quad (5.10)$$

Let u generate $(1/\lambda)\text{sta}_{\mathbf{L}}v_1$. Hence, $\langle u \rangle \cong \mathbb{Z}_\lambda$, and $\text{sta}_{\mathbf{L}}v_1 = \{1\} \cup \{\lambda u^j | 1 \leq j \leq \lambda - 1\}$. Then (5.8) and (5.9) yield

$$\begin{aligned} v_i v_{\lambda-i} &= \lambda \left(\sum_{j=0}^{\lambda-1} u^j \right), & \text{for all } i; \\ v_i u^j &= v_i, & \text{for all } i, j. \end{aligned} \quad (5.11)$$

Since $\text{Supp}_{\mathbf{B}}(b\bar{b}) \cap \mathbf{L} = \text{Supp}_{\mathbf{B}}(b^2) \cap \mathbf{L} = \{1\}$, Lemma 5.1 implies that $\text{sta}_{\mathbf{L}}b = \{1\}$. Thus, $\{bu^j | 0 \leq j \leq \lambda - 1\}$ is a set of λ distinct elements of $\mathbf{L}^{(3)} \setminus \mathbf{L}^{(2)}$. Since $(bv_i, b) = (v_i, b^2) > 0$, we have that $b \in \text{Supp}_{\mathbf{B}}(bv_i)$ and $bu^j \in \text{Supp}_{\mathbf{B}}(bv_i u^j)$ for all j . But $bv_i u^j = bv_i$, by (5.9), so it follows that

$$bv_i = b \left(\sum_{j=0}^{\lambda-1} u^j \right), \quad \text{for all } i. \quad (5.12)$$

It now follows from (5.5), (5.10), (5.11), and (5.12) that

$$\mathbf{B} = \mathbf{B}_b = \{1, \lambda u, \lambda u^2, \dots, \lambda u^{\lambda-1}, v_1, v_2, \dots, v_{\lambda-1}, b, bu, bu^2, \dots, bu^{\lambda-1}\},$$

and $\mathbf{B} \cong_x \mathbf{N}((3), \lambda)$.

Finally, we suppose that $n \geq 4$ and work toward a contradiction. Set $b^2 = \lambda 1 + \sum_{i=1}^{\lambda-1} b_i$, for some $b_i \in \mathbf{B} \setminus \{1\}$. By the previous case, $\mathbf{B}/\mathbf{L}^{(n-3)} \cong_x \mathbf{N}((3), \lambda)$. Let $e = e_{\mathbf{L}^{(n-3)}}$. Then

$$(be)^2 = \lambda e + \sum_{i=1}^{\lambda-1} b_i e,$$

where the $b_i e$ are $\lambda - 1$ distinct, standard elements of $\mathbf{B}/\mathbf{L}^{(n-3)}$. Also, $\mathbf{L}^{(n-2)}/\mathbf{L}^{(n-3)} \cong_x \mathbb{Z}'_\lambda$.

Let ue denote a rescaling of degree 1 of a generator of $\mathbf{L}^{(n-2)}/\mathbf{L}^{(n-3)}$. Thus, $(ue)^2$ also generates $(1/\lambda)\mathbf{L}^{(n-2)}/\mathbf{L}^{(n-3)}$, since λ is odd. Now

$$((be)(ue))^2 = \lambda(ue)^2 + \sum_{i=1}^{\lambda-1} (b_i e)(ue)^2,$$

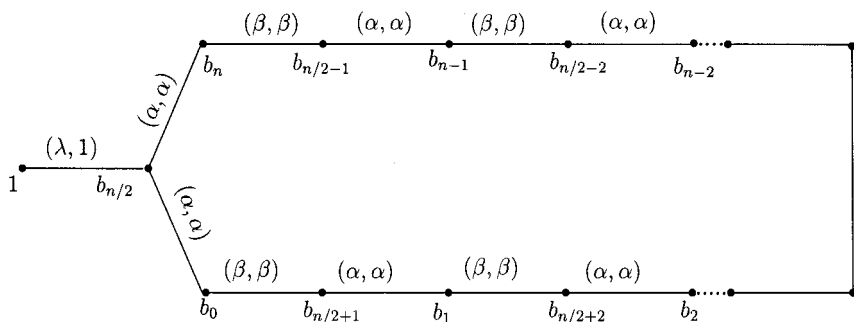
where $\lambda(ue)^2$ is a nontrivial linear element of $\mathbf{B}/\mathbf{L}^{(n-3)}$, $(be)(ue) \in \mathbf{B}/\mathbf{L}^{(n-3)}$, and the $(b_i e)(ue)^2$ form a set of $\lambda - 1$ distinct nonlinear elements of $\mathbf{B}/\mathbf{L}^{(n-3)}$. Let $c \in \mathbf{B}$ such that $ce = (be)(ue)$. Then

$$c^2 e = \lambda(ue)^2 + \sum_{i=1}^{\lambda-1} (b_i e)(ue)^2$$

implies that $c^2 = \sum_{i=0}^{\lambda-1} c_i$, where $c_0 e = \lambda(ue)^2$ and $c_i e = (b_i e)(ue)^2$ for $i > 0$. Since $c_0 \notin \mathbf{L}^{(n-3)} \supseteq \mathbf{L}$ (as $n \geq 4$), it follows that all the c_i are distinct and standard. This contradicts Lemma 5.4 and proves the theorem.

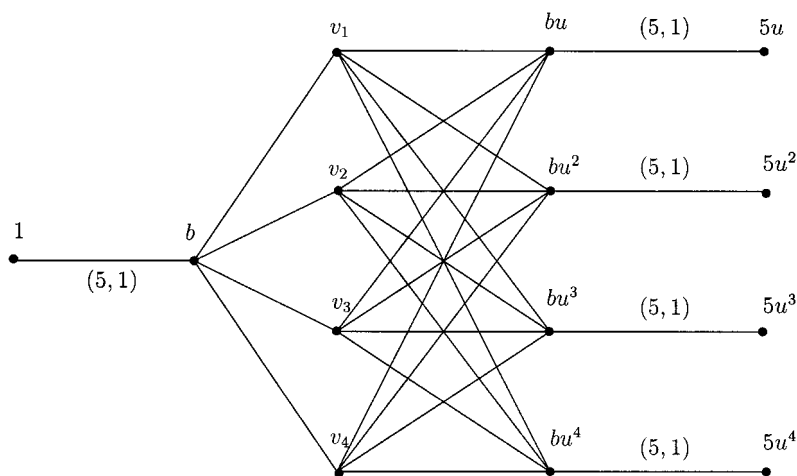
6. REPRESENTATION GRAPHS

Let (A, \mathbf{B}) be a table algebra and let $b = b_i$ be a real element of \mathbf{B} . Then the representation graph $\Gamma_b(\mathbf{B})$, as defined in [B2, Sect. 5], is an undirected, labeled graph with vertex set in bijection with \mathbf{B} . An edge joins vertices b_i and b_j iff $\beta_{ij} \neq 0$ (iff $\beta_{ji} \neq 0$), and is labeled by the ordered pair (β_{ij}, β_{ji}) . Figure 6.1 displays the graph when $\mathbf{B} = \mathbf{T}_n(\lambda)$ for n even and $b = b_{n/2}$. Here, $\alpha = (\lambda - 1)/2$ and $\beta = (\lambda + 1)/2$ as in Example 3.3. The graph for $\mathbf{B} = \mathbf{N}((3), 5)$, with b as in Example 1.26, is given in Fig. 6.2. All unspecified labelings in Fig. 6.2 are $(1, 1)$.



$\Gamma_{b_{n/2}}(\mathbf{T}_n(\lambda))$ (n even)

FIGURE 6.1.



$$\Gamma_b(\mathbf{N}((3), 5))$$

FIGURE 6.2.

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